

Yangian symmetry of boundary scattering in
AdS/CFT
and the explicit form of bound state reflection
matrices

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Abstract

The reflection matrices of multi magnon bound states are obtained explicitly by exploiting the Yangian symmetry of boundary scattering on the $Y = 0$ maximal giant graviton brane.

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1 Introduction

Since the fundamental two particle S matrix of AdS/CFT [1, 2, 3] is determined by the symmetries (centrally extended $su(2|2)$ ¹) up to an overall scalar factor the discovery that it also admits an interesting Yangian symmetry $Y(su(2|2))$ [4] may seem to be relevant only from a mathematical point of view. (Further investigations of Yangian symmetry in AdS/CFT can be found e.g. in [5]-[8]). The power of the Yangian symmetry becomes manifest when one tries to construct the S matrices for the infinite tower of magnon bound states [9, 10]: ordinary symmetry considerations alone are not sufficient to fix the form of the bound state S matrix elements in general [10] and only the Yangian symmetry is powerful enough to do this in case of the Q bound state - Q' bound state scatterings ($Q, Q' \geq 2$) [11].

The first steps to extend the Yangian symmetry to scattering in *boundary* AdS/CFT are made in [12, 13]. In [12] it is shown that the remaining symmetry algebra of boundary scattering on the $Y = 0$ brane is not restrictive enough to make the reflection matrix of the $Q = 2$ bound state diagonal and leaves some elements of this matrix undetermined in contrast to the reflection of the fundamental magnon, where the reflection matrix is diagonal and is determined up to an overall factor. To remedy this situation the authors of [12] construct a conserved charge corresponding to a generator of the boundary remnant of the bulk Yangian symmetry and show that this extra conservation gives the missing equations leading to a solution of the $Q = 2$ bound state reflection matrix.

In [13] the structure of the Yangian symmetry of the $Y = 0$ brane is described in details, building on a series of earlier papers [14]-[16]. Using the superspace formalism introduced in [10] they also construct the reflection matrices of the fundamental ($Q = 1$) and $Q = 2$ bound state reflections in terms of appropriate differential operators acting on superspace and combining this with the Yangian symmetry obtain the explicit form of these R matrices.

The aim of this paper is to consider the reflection of a general Q magnon bound state ($Q > 2$) on the $Y = 0$ brane and to show that the boundary remnant of the Yangian symmetry is powerful enough to yield an explicit solution even in this case.

The paper is organized as follows: in the second section we review the superspace description of the Q magnon bound states and discuss the general structure of their reflection matrices with the outcome that $Q - 1$ functions in those matrices remain undetermined by ordinary symmetry considerations. In section 3 we describe briefly the Yangian symmetry of the $Y = 0$ brane and derive the explicit solution for reflection matrices of the Q magnon bound states. In section 4 we summarize our results and

¹In this paper we focus on one copy of $su(2|2)$ symmetric S and R matrices, in the full AdS/CFT the S and R matrices are tensor products of two such copies.

discuss also briefly the question of Yangian symmetry for the bound states of the *mirror* model [17].

2 Magnon bound states and the structure of their reflection matrices

In this section we collect the necessary ingredients to describe the bound states of Q fundametal magnons as well as derive the general structure of the reflection matrices describing the reflections of these bound states on the $Y = 0$ brane.

2.1 Q magnon bound state representation

The centrally extended $su(2|2)$ algebra consists of the rotation generators \mathbb{L}_a^b , \mathbb{R}_α^β , the supersymmetry generators \mathbb{Q}_α^a , $\mathbb{Q}_a^{\dagger\alpha}$, and the central elements \mathbb{C} , \mathbb{C}^\dagger , \mathbb{H} . Latin indices a, b, \dots take values $\{1, 2\}$, while Greek indices α, β, \dots take values $\{3, 4\}$. These generators have the following nontrivial commutation relations [2, 3]

$$\begin{aligned}
[\mathbb{L}_a^b, \mathbb{J}_c] &= \delta_c^b \mathbb{J}_a - \frac{1}{2} \delta_a^b \mathbb{J}_c, & [\mathbb{R}_\alpha^\beta, \mathbb{J}_\gamma] &= \delta_\gamma^\beta \mathbb{J}_\alpha - \frac{1}{2} \delta_\alpha^\beta \mathbb{J}_\gamma, \\
[\mathbb{L}_a^b, \mathbb{J}^c] &= -\delta_a^c \mathbb{J}^b + \frac{1}{2} \delta_a^b \mathbb{J}^c, & [\mathbb{R}_\alpha^\beta, \mathbb{J}^\gamma] &= -\delta_\alpha^\gamma \mathbb{J}^\beta + \frac{1}{2} \delta_\alpha^\beta \mathbb{J}^\gamma, \\
\{\mathbb{Q}_\alpha^a, \mathbb{Q}_\beta^b\} &= \epsilon_{\alpha\beta} \epsilon^{ab} \mathbb{C}, & \{\mathbb{Q}_a^{\dagger\alpha}, \mathbb{Q}_b^{\dagger\beta}\} &= \epsilon^{\alpha\beta} \epsilon_{ab} \mathbb{C}^\dagger, \\
\{\mathbb{Q}_\alpha^a, \mathbb{Q}_b^{\dagger\beta}\} &= \delta_b^a \mathbb{R}_\alpha^\beta + \delta_\alpha^\beta \mathbb{L}_b^a + \frac{1}{2} \delta_b^a \delta_\alpha^\beta \mathbb{H}, & &
\end{aligned} \tag{2.1}$$

where \mathbb{J}_i (\mathbb{J}^i) denotes any lower (upper) index of a generator, respectively.

The Q magnon bound states form a $4Q$ -dimensional atypical totally symmetric representation of $su(2|2)$ [9, 10]. Using the convenient superspace formalism of [10], the $su(2|2)$ generators can be represented by differential operators acting on a vector space of polynomials built from two bosonic (w_a) and two fermionic (θ_α) variables, as follows:

$$\begin{aligned}
\mathbb{L}_a^b &= w_a \frac{\partial}{\partial w_b} - \frac{1}{2} \delta_a^b w_c \frac{\partial}{\partial w_c}, & \mathbb{R}_\alpha^\beta &= \theta_\alpha \frac{\partial}{\partial \theta_\beta} - \frac{1}{2} \delta_\alpha^\beta \theta_\gamma \frac{\partial}{\partial \theta_\gamma}, \\
\mathbb{Q}_\alpha^c &= a \theta_\alpha \frac{\partial}{\partial w_c} + b \epsilon^{cb} \epsilon_{\alpha\beta} w_b \frac{\partial}{\partial \theta_\beta}, & \mathbb{Q}_a^{\dagger\alpha} &= d w_a \frac{\partial}{\partial \theta_\alpha} + c \epsilon_{ab} \epsilon^{\alpha\beta} \theta_\beta \frac{\partial}{\partial w_b}, \\
\mathbb{C} &= ab \left(w_a \frac{\partial}{\partial w_a} + \theta_\alpha \frac{\partial}{\partial \theta_\alpha} \right), & \mathbb{C}^\dagger &= cd \left(w_a \frac{\partial}{\partial w_a} + \theta_\alpha \frac{\partial}{\partial \theta_\alpha} \right), \\
\mathbb{H} &= (ad + bc) \left(w_a \frac{\partial}{\partial w_a} + \theta_\alpha \frac{\partial}{\partial \theta_\alpha} \right). & &
\end{aligned} \tag{2.2}$$

The a, b, c, d parameters (satisfying $ad - bc = 1$) are functions of the Q magnon bound state's momentum p [2, 10]:

$$a = \sqrt{\frac{g}{2Q}}\eta, \quad b = \sqrt{\frac{g}{2Q}}\frac{i}{\eta}\left(\frac{x^+}{x^-} - 1\right), \quad c = -\sqrt{\frac{g}{2Q}}\frac{\eta}{x^+}, \quad d = \sqrt{\frac{g}{2Q}}\frac{x^+}{i\eta}\left(1 - \frac{x^-}{x^+}\right) \quad (2.3)$$

where (we follow the phase convention of [10])

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{2Qi}{g}, \quad \frac{x^+}{x^-} = e^{ip}, \quad \eta = e^{ip/4}\sqrt{i(x^- - x^+)}. \quad (2.4)$$

We decompose the $4Q$ dimensional representation space, $\mathcal{V}^Q(p)$, as $4Q = (Q+1) + (Q-1) + Q + Q$ and parameterize the sub-spaces as

$$\begin{aligned} Q+1 \rightarrow |j\rangle^1 &= \frac{w_1^{Q-j}w_2^j}{\sqrt{(Q-j)!j!}}, & j = 0, \dots, Q \\ Q-1 \rightarrow |j\rangle^2 &= \frac{w_1^{Q-2-j}w_2^j}{\sqrt{(Q-2-j)!j!}}\theta_3\theta_4, & j = 0, \dots, Q-2 \\ Q \rightarrow |j\rangle^3 &= \frac{w_1^{Q-1-j}w_2^j}{\sqrt{(Q-1-j)!j!}}\theta_3, & j = 0, \dots, Q-1 \\ Q \rightarrow |j\rangle^4 &= \frac{w_1^{Q-1-j}w_2^j}{\sqrt{(Q-1-j)!j!}}\theta_4, & j = 0, \dots, Q-1. \end{aligned} \quad (2.5)$$

(Note, that for the fundamental magnon, $Q = 1$, the second subspace is absent and the 4 dimensional representation is parameterized as $4 = 2 + 1 + 1$ with $|0\rangle^1 = w_1$, $|1\rangle^1 = w_2$, $|0\rangle^3 = \theta_3$, $|0\rangle^4 = \theta_4$).

2.2 The structure of the reflection matrix

Next we investigate to what extent the remaining symmetries of the $Y = 0$ brane restrict the reflection matrix of the Q magnon bound states. These remaining symmetries form an $su(2|1)$ sub-algebra [18], consisting of the generators

$$\mathbb{L}_1^1, \quad \mathbb{L}_2^2, \quad \mathbb{H}, \quad \mathbb{R}_\alpha^\beta, \quad \mathbb{Q}_\alpha^1, \quad \mathbb{Q}_1^{\dagger\alpha}. \quad (2.6)$$

To describe the reflection matrix we follow [12], [13] and define a boundary vacuum state $|0\rangle_B$ corresponding to a trivial vector space $\mathcal{V}(0)$ annihilated by all $su(2|1)$ generators. This makes it possible to define the (super-space) R -matrix for the reflection of bulk magnon bound states as an operator acting on the tensor product spaces

$$R(p) : \quad \mathcal{V}^Q(p) \otimes \mathcal{V}(0) \rightarrow \mathcal{V}^Q(-p) \otimes \mathcal{V}(0) \quad (2.7)$$

where the reflection matrix is given as a differential operator

$$R(p) = \sum_i r_i(p) \Lambda_i \quad (2.8)$$

acting on the super-space. Here Λ_i span a basis of invariant differential operators built from w_a , and θ_a . Since all \mathbb{J}^i generators of $su(2|1)$, eq.(2.6), annihilate $|0\rangle_B$, on the tensor product space they have the coproducts [13]:

$$\Delta(\mathbb{J}^i) = \mathbb{J}^i \otimes 1.$$

As a consequence requiring the reflections to respect the $su(2|1)$ symmetry amounts to imposing the vanishing of the commutator $[\mathbb{J}^i, R]|j\rangle^I$ $I = a, \alpha$.

Since \mathbb{L}_1^1 ($\equiv -\mathbb{L}_2^2$) acts diagonally on the various sub-spaces

$$\mathbb{L}_1^1|j\rangle^1 = \frac{1}{2}(Q-2j)|j\rangle^1, \quad \mathbb{L}_1^1|j\rangle^2 = \frac{1}{2}(Q-2-2j)|j\rangle^2, \quad \mathbb{L}_1^1|j\rangle^\alpha = \frac{1}{2}(Q-1-2j)|j\rangle^\alpha,$$

and \mathbb{R}_α^β acts non trivially only for $|j\rangle^\gamma$, and in particular

$$\mathbb{R}_4^3|j\rangle^3 = |j\rangle^4, \quad \mathbb{R}_3^4|j\rangle^4 = |j\rangle^3,$$

requiring these ‘‘bosonic’’ symmetry generators to commute with R restricts the form of the reflection matrix as

$$R = \sum_{l=0}^Q A_l \Lambda_{(1)}^l + \sum_{l=0}^{Q-2} B_l \Lambda_{(2)}^l + \sum_{l=0}^{Q-1} C_l \Lambda_{(3)}^l + \sum_{l=0}^{Q-2} D_l \Lambda_{(4)}^l + \sum_{l=0}^{Q-2} E_l \Lambda_{(5)}^l. \quad (2.9)$$

Here the various differential operators are given as

$$\Lambda_{(1)}^l = \frac{w_1^{Q-l} w_2^l}{(Q-l)!l!} \frac{\partial^Q}{\partial w_1^{Q-l} \partial w_2^l}, \quad \Lambda_{(2)}^l = \frac{w_1^{Q-2-l} w_2^l}{(Q-2-l)!l!} \theta_3 \theta_4 \frac{\partial^{Q-2}}{\partial w_1^{Q-2-l} \partial w_2^l} \frac{\partial^2}{\partial \theta_4 \partial \theta_3},$$

$$\Lambda_{(3)}^l = \frac{w_1^{Q-1-l} w_2^l}{(Q-1-l)!l!} \frac{\partial^{Q-1}}{\partial w_1^{Q-1-l} \partial w_2^l} \theta_\alpha \frac{\partial}{\partial \theta_\alpha},$$

$$\Lambda_{(4)}^l = \frac{w_1^{Q-2-l} w_2^l}{(Q-2-l)!l!} \theta_3 \theta_4 \frac{\partial^Q}{\partial w_1^{Q-1-l} \partial w_2^{l+1}}, \quad \Lambda_{(5)}^l = \frac{w_1^{Q-1-l} w_2^{l+1}}{(Q-2-l)!l!} \frac{\partial^{Q-2}}{\partial w_1^{Q-2-l} \partial w_2^l} \frac{\partial^2}{\partial \theta_4 \partial \theta_3},$$

and A_l , B_l , C_l , D_l and E_l are $5Q-2$ (unknown) functions of p . (For $Q=1$ the second, fourth and fifth sums are missing leaving only A_0 , A_1 and C_0 to be determined). The first three sums in (2.9) describe diagonal reflections while the fourth and fifth ones describe off-diagonal reflections between multi-magnon polarizations in the $Q+1$ and $Q-1$ subspaces: this possibility arises as a result of the coinciding eigenvalues of the bosonic symmetry generators in the two subspaces.

To obtain equations for the unknown functions next we consider the restrictions following from requiring also the fermionic generators to commute with reflections. For this we list the action of the fermionic generators on the various sub-spaces

$$\begin{aligned}
\mathbb{Q}_\alpha^1 |j\rangle^1 &= a\sqrt{Q-j}|j\rangle^\alpha, & \mathbb{Q}_\alpha^1 |j\rangle^2 &= -b\sqrt{j+1}|j+1\rangle^\alpha, \\
\mathbb{Q}_\alpha^1 |j\rangle^\beta &= \epsilon^{\alpha\beta}(a\sqrt{Q-1-j}|j\rangle^2 + b\sqrt{j+1}|j+1\rangle^1), \\
\mathbb{Q}_1^{\dagger\alpha} |j\rangle^1 &= c\sqrt{j}\epsilon^{\alpha\beta}|j-1\rangle^\beta, & \mathbb{Q}_1^{\dagger\alpha} |j\rangle^2 &= d\sqrt{Q-1-j}\epsilon^{\alpha\beta}|j\rangle^\beta, \\
\mathbb{Q}_1^{\dagger\alpha} |j\rangle^\beta &= \delta^{\alpha\beta}(d\sqrt{Q-j}|j\rangle^1 - c\sqrt{j}|j-1\rangle^2).
\end{aligned}$$

To make the subsequent equations simpler we introduce the notation $\dot{f} \equiv f(-p)$ for any function $f(p)$. Using this and recalling (2.7) the action of the reflection matrix can be written as

$$\begin{aligned}
R|j\rangle^1 &= A_j(|\dot{j}\rangle^1) + \sqrt{(Q-j)\dot{j}}D_{j-1}(|\dot{j}-1\rangle^2), & j &= 0, \dots, Q \\
R|j\rangle^2 &= B_j(|\dot{j}\rangle^2) + \sqrt{(Q-1-j)(j+1)}E_j(|\dot{j}+1\rangle^1), & j &= 0, \dots, Q-2 \\
R|j\rangle^\alpha &= C_j(|\dot{j}\rangle^\alpha), & j &= 0, \dots, Q-1
\end{aligned}$$

The vanishing of the commutator on the four subspaces $[\mathbb{Q}_\alpha^1, R]|j\rangle^I$ $I = a, \alpha$ leads to the following equations:

$$\begin{aligned}
aC_j &= \dot{a}A_j - \dot{b}jD_{j-1}, & j &= 0, \dots, Q-1, \\
bC_{j+1} &= \dot{b}B_j - \dot{a}(Q-1-j)E_j, & j &= 0, \dots, Q-2, \\
\dot{a}C_j &= aB_j + b(j+1)D_j, & j &= 0, \dots, Q-2, \\
\dot{b}C_j &= bA_{j+1} + a(Q-1-j)E_j, & j &= 0, \dots, Q-1,
\end{aligned} \tag{2.10}$$

while using $\mathbb{Q}_1^{\dagger\alpha}$ instead of \mathbb{Q}_α^1 gives

$$\begin{aligned}
cC_{j-1} &= \dot{c}A_j + \dot{d}(Q-j)D_{j-1}, & j &= 1, \dots, Q, \\
dC_j &= \dot{d}B_j + \dot{c}(1+j)E_j, & j &= 0, \dots, Q-2, \\
\dot{d}C_j &= dA_j - cjE_{j-1}, & j &= 0, \dots, Q-1, \\
\dot{c}C_j &= cB_{j-1} - d(Q-j)D_{j-1}, & j &= 1, \dots, Q-1.
\end{aligned} \tag{2.11}$$

In both sets of equations there are altogether $4Q-2$ equations. However it is straightforward to show that in both sets there are $2(Q-1)$ relations between these equations, leaving in both sets $2Q$ independent equations. Since $\dot{a}/a = d/\dot{d} = e^{-ip/2}$ and $\dot{c}/c = b/\dot{b} = -e^{ip/2}$ the two (independent) “corner” equations connecting A_0 to C_0 and

A_Q to C_{Q-1} are identical in (2.10) and in (2.11); thus in the two sets there are only $4Q - 2$ independent equations. This means that Q of the $5Q - 2$ unknown functions is not determined by requiring the symmetry transformations and reflections to commute. On physical grounds we expect that one overall scalar factor in the reflection matrix is determined by consideration going beyond the symmetries (unitarity, crossing, fusion etc.); setting this overall scale to one still leaves $Q - 1$ functions undetermined. In this respect the reflection of Q magnon bound states is different from the reflection of fundamental magnons as was discovered in the $Q = 2$ case in [12]. In the next section we show that invoking the Yangian extension of the $su(2|1)$ symmetry provides the necessary extra equations even in the general case.

3 Yangian symmetry and the explicit form of the reflection matrix

In this section – following [12] [13] – we describe in a nutshell the Yangian symmetry of the $Y = 0$ brane and present the explicit solution for the reflection matrix of the Q magnon bound state.

3.1 The Yangian of the $Y = 0$ brane

The Yangian extension ($Y(\mathfrak{g})$) of a *bulk* Lie symmetry \mathfrak{g} is a deformation of the universal enveloping algebra of the polynomial algebra $\mathfrak{g}(u)$. It is generated by grade-1 “Yangian” generators $\hat{\mathbb{J}}^A$ besides the grade-0 generators \mathbb{J}^A of \mathfrak{g} . Their commutation relations have the form

$$[\mathbb{J}^A, \mathbb{J}^B] = f^{AB}{}_C \mathbb{J}^C, \quad [\mathbb{J}^A, \hat{\mathbb{J}}^B] = f^{AB}{}_C \hat{\mathbb{J}}^C,$$

and must obey the Jacobi and Serre relations. To satisfy these in case of $\mathfrak{g} = su(2|2)$ is not entirely straightforward as the Killing form of $su(2|2)$ is degenerate, but this problem may be circumvented and the explicit form of $Y(su(2|2))$ - together with the coproducts of the Yangian generators - is known [4]. (The coproducts are necessary to explore the action of Y on two particle states).

In constructing finite dimensional representations of $Y(\mathfrak{g})$ a crucial role is played by the “evaluation representation”, where the grade-1 generators have the form

$$\hat{\mathbb{J}}^A |u\rangle = -i \frac{g}{2} u \mathbb{J}^A |u\rangle.$$

It is shown in [4] [11] that the (multi) magnon (bound) states are of this form, where

$$u \equiv u(p) = \frac{1}{2} \left(x^+ + \frac{1}{x^+} + x^- + \frac{1}{x^-} \right), \quad (3.1)$$

and x^\pm are defined in (2.4). In [4] it is shown that the fundamental magnon's S -matrix is constrained up to an overall phase by imposing the invariance of the S -matrix under (the coproducts of) $Y(su(2|2))$. In the case of scattering of a Q bound state and a Q' bound state ($Q, Q' \geq 2$) the $su(2|2)$ symmetry algebra is not enough to fix all elements of S ; the necessary additional constraints may be obtained from the Yangian symmetry [11].

As discussed in a series of papers [14], [15], [16] only a remnant (denoted as $Y(\mathbf{h}, \mathbf{g})$) of the bulk Yangian symmetry survives when the bulk theory is restricted by a boundary, which although preserving integrability preserves only a sub-algebra $\mathbf{h} \subset \mathbf{g}$ of the symmetry of the bulk fields. As turns out (\mathbf{h}, \mathbf{g}) must form a symmetric pair

$$\mathbf{g} = \mathbf{h} + \mathbf{m}, \quad [\mathbf{h}, \mathbf{h}] \subset \mathbf{h}, \quad [\mathbf{h}, \mathbf{m}] \subset \mathbf{m}, \quad [\mathbf{m}, \mathbf{m}] \subset \mathbf{h},$$

and $Y(\mathbf{h}, \mathbf{g})$ is generated by $(\mathbb{J}^i, \tilde{\mathbb{J}}^p)$, where $i(j, k)$ run over the \mathbf{h} indices and $p(q, r)$ over the \mathbf{m} indices, and

$$\tilde{\mathbb{J}}^p = \hat{\mathbb{J}}^p + \frac{1}{2} f_{qi}^p \mathbb{J}^q \mathbb{J}^i.$$

(The extra twisting represented by the second term is necessary to guarantee that products of bulk and boundary states still represent $Y(\mathbf{h}, \mathbf{g})$).

In case of the $Y = 0$ brane the generators of \mathbf{h} are given in (2.6) and the subspace \mathbf{m} is generated by

$$\mathbb{L}_2^1, \quad \mathbb{L}_1^2, \quad \mathbb{Q}_\gamma^2, \quad \mathbb{Q}_2^{\dagger\gamma}, \quad \mathbb{C}, \quad \mathbb{C}^\dagger.$$

(It is straightforward to check that they indeed form a symmetric pair). Now one can readily construct the Yangian generators $\tilde{\mathbb{J}}^p$ and their coproducts exploiting that all symmetry generators annihilate the boundary vacuum $|0\rangle_B$; their explicit form is given in [13]. In this paper we use only one of them,

$$\tilde{\mathbb{Q}} \otimes 1 \equiv \Delta \tilde{\mathbb{L}}_2^1 = \left(\hat{\mathbb{L}}_2^1 + \frac{1}{2} \left(\mathbb{L}_2^1 \mathbb{L}_1^1 - \mathbb{L}_2^1 \mathbb{L}_2^2 - \mathbb{Q}_2^{\dagger\gamma} \mathbb{Q}_\gamma^1 \right) \right) \otimes 1, \quad (3.2)$$

the same one introduced in [12].

3.2 The explicit form of the reflection matrix

We obtain equations supplementing (2.10, 2.11) by imposing the vanishing of the commutator between $\tilde{\mathbb{Q}}$ and R . To implement these we need the action of $\tilde{\mathbb{Q}}$ on the various

subspaces:

$$\begin{aligned}
\tilde{\mathbb{Q}}|j\rangle^1 &= \sqrt{(Q-j)(j+1)} \left(-i\frac{g}{2}u + \frac{Q}{2} - j - ad \right) |j+1\rangle^1 - \sqrt{(Q-j)(Q-1-j)ac} |j\rangle^2, \\
\tilde{\mathbb{Q}}|j\rangle^2 &= \sqrt{(Q-2-j)(j+1)} \left(-i\frac{g}{2}u + \frac{Q-2}{2} - j + bc \right) |j+1\rangle^2 \\
&\quad + \sqrt{(j+1)(j+2)bd} |j+2\rangle^1, \\
\tilde{\mathbb{Q}}|j\rangle^\beta &= \sqrt{(Q-1-j)(j+1)} \left(-i\frac{g}{2}u + \frac{Q-1}{2} - j - \frac{1}{2} \right) |j+1\rangle^\beta.
\end{aligned} \tag{3.3}$$

Since both R and $\tilde{\mathbb{Q}}$ act diagonally on the fermionic subspaces $|j\rangle^\alpha$ requiring $[\tilde{\mathbb{Q}}, R]|j\rangle^\gamma = 0$ gives $Q-1$ equations

$$C_{j+1} = \Phi(j)C_j, \quad j = 0, \dots, Q-2, \quad \text{where} \quad \Phi(j) = \frac{i\frac{g}{2}u + \frac{Q}{2} - j - 1}{-i\frac{g}{2}u + \frac{Q}{2} - j - 1}. \tag{3.4}$$

These equations determine all the C_j in terms of C_0 :

$$C_{j+1} = C_0 \prod_{l=0}^j \Phi(l), \quad j = 0, \dots, Q-2. \tag{3.5}$$

The structure of the set of solutions $\{C_0, C_1, \dots, C_{Q-1}\}$ depends on whether Q is even or odd. For Q even, $l_0 = \frac{Q-2}{2}$ is integer, and exploiting

$$\Phi(Q-2-l) = \frac{1}{\Phi(l)}, \quad \text{and} \quad \Phi(l_0) = -1,$$

one can show that the set of C -s has the form

$$\{C_0, C_1, \dots, C_{l_0}, -C_{l_0}, \dots, -C_1, -C_0\}; \tag{3.6}$$

while for Q odd, $Q = 2r + 1$, exploiting

$$\Phi(r-k) = \frac{1}{\Phi(r+k-1)}, \quad k = 1, \dots, r$$

one obtains that the set of C -s have the form

$$\{C_0, C_1, \dots, C_{r-1}, C_r, C_{r-1}, \dots, C_1, C_0\}. \tag{3.7}$$

For $Q = 2$, $l_0 = 0$, and (3.6) gives $C_1 = -C_0$, which is consistent with [12].

Now one can use these explicitly known C_j -s in eq.(2.10, 2.11) to determine the remaining unknown functions. We choose the normalization $A_0 = 1$ (since $|0\rangle^1$ has the highest \mathbb{L}_1^1 value), using this in the ‘‘corner equations’’ leads to

$$C_0 = A_0 \frac{d}{d} = e^{-ip/2}, \quad A_Q = \frac{c}{c} C_{Q-1} = e^{-ip} \prod_{l=0}^{Q-2} \Phi(l) = (-1)^Q e^{-ip}. \tag{3.8}$$

The other unknown coefficients in the reflection matrix are found to be given by

$$\begin{aligned}
A_{j+1} &= \left(\prod_{l=0}^{j-1} \Phi(l) \right) \frac{(Q-1-j)\Phi(j)x^+ - (j+1)/x^+}{(Q-1-j)x^+ + (j+1)/x^-}, \\
B_j &= \left(\prod_{l=0}^{j-1} \Phi(l) \right) \frac{(Q-1-j)x^- - (j+1)\Phi(j)/x^-}{(Q-1-j)x^+ + (j+1)/x^-}, \\
E_j &= -D_j = e^{-ip/2} \left(\prod_{l=0}^{j-1} \Phi(l) \right) \frac{x^+\Phi(j) + x^-}{x^+x^-(Q-1-j) + (j+1)}, \quad j = 0, \dots, Q-2,
\end{aligned} \tag{3.9}$$

where, for $j = 0$, $\prod_{l=0}^{j-1} \Phi(l) = 1$ is understood. The expressions appearing in eq.(3.5, 3.8) and (3.9) constitute the explicit form of functions defining the reflection matrix of the Q magnon bound state (apart from the overall scalar factor) and they represent the main result of this paper.

For $Q = 2$ this solution has the form: $A_0 = 1$ and

$$\begin{aligned}
C_0 = e^{-ip/2} = -C_1, \quad A_2 = e^{-ip}, \quad A_1 = -\frac{x^+ + \frac{1}{x^+}}{x^+ + \frac{1}{x^-}}, \\
B_0 = \frac{x^- + \frac{1}{x^-}}{x^+ + \frac{1}{x^-}}, \quad E_0 = -D_0 = e^{-ip/2} \frac{x^- - x^+}{1 + x^+x^-},
\end{aligned}$$

which, recalling eq.(2.9), is nothing but $R_{AN}(-p) \equiv R_{AN}^{-1}(p)$, with R_{AN} being the reflection matrix found in [12]. (This difference follows from the difference between our definition of the reflection matrix (2.7), and eq.(3.10) in [12]).

It is straightforward to check that the solution given by eq.(3.5, 3.8) and (3.9) satisfies the unitarity constraint

$$R(-p)R(p) = 1. \tag{3.10}$$

Indeed the “diagonal” part of the reflection matrix -i.e. the one determined by A_0 , A_Q and C_j satisfy this as a consequence of $\Phi(l) = 1/\Phi(l)$, while on $|j\rangle^1$, $|j\rangle^2$ (3.10) gives four equations between A_{j+1} , D_j , E_j , B_j and their “dotted” versions, which are found to be satisfied.

We used only the fermionic sub-space component of the commutator $[\tilde{Q}, R]|j\rangle^\gamma$ to determine the functions characterizing the reflection matrix. For consistency $[\tilde{Q}, R]|j\rangle^1$ and $[\tilde{Q}, R]|j\rangle^2$ should also vanish. It is straightforward to write explicitly the equations these requirements impose on A_{j+1} , D_j , E_j , B_j and we checked (albeit sometimes only numerically) that the solution given by eq.(3.5, 3.8) and (3.9) satisfies all of them.

The eventual verification of our solution is to show that it solves the boundary Yang-Baxter equation. Denoting by R_Q ($R_{Q'}$) the reflection matrix of the Q (Q') magnon

bound states and by $S_{QQ'}$ their bulk S -matrix, the boundary Yang-Baxter equation has the schematic form:

$$R_Q(p)S_{Q'Q}(q, -p)R_{Q'}(q)S_{QQ'}(-p, -q) = S_{Q'Q}(q, p)R_{Q'}(q)S_{QQ'}(p, -q)R_Q(p). \quad (3.11)$$

Although the explicit form of $S_{QQ'}$ is known [11], because of the complexity of these expressions, the verification of (3.11) is beyond the scope of the present paper.

For physical applications one needs the explicit form of the overall scalar factor we ignored so far. In fact this scalar factor is known: for the fundamental magnon, $Q = 1$, an equation for this factor is derived from unitarity and crossing considerations in [18], this equation is solved in [19]; finally using this solution as input the scalar factor of the Q magnon bound state is determined by the fusion method in [20] exploiting that the bound state's component with the highest value of \mathbb{L}_1^1 scatters and reflects diagonally.

4 Summary and discussion

In this paper the reflection of multi magnon bound states on the $Y = 0$ maximal giant graviton brane is investigated. It is shown that the reflection matrices of the Q magnon bound states can be described as appropriate linear combinations of projectors and are characterized in terms of $5Q - 2$ unknown functions. $Q - 1$ of these functions remain undetermined by ordinary symmetry considerations, i.e. by requiring the surviving $su(2|1)$ generators to commute with reflections. Invoking the Yangian extension of the $su(2|1)$ symmetry of the $Y = 0$ brane solves the problem: requiring the same Yangian generator that is used in [12] to commute with reflections gives precisely $Q - 1$ additional equations such that the whole system admits a consistent solution. These explicit reflection matrices - when augmented by the overall scalar factor available in the literature [20] - can be used as starting points to analyze the boundary finite size effects for magnon bound states.

In [13] also a “toy model” boundary is discussed together with its Yangian symmetry. In this model the hypothetical boundary breaks $su(2|2)$ down to $\tilde{\mathbf{h}} = su(1|2)$ generated by

$$\mathbb{R}_3^3 = \frac{1}{2}(\theta_3 \frac{\partial}{\partial \theta_3} - \theta_4 \frac{\partial}{\partial \theta_4}), \quad \mathbb{L}_a^b, \quad \mathbb{Q}_3^a, \quad \mathbb{Q}_a^{\dagger 3}, \quad \mathbb{H}. \quad (4.1)$$

In [13] it is shown that the corresponding Yangian extension $Y(\tilde{\mathbf{h}}, su(2|2))$ is in a certain sense redundant or trivial, since imposing only the $\tilde{\mathbf{h}}$ symmetry on the reflection matrix R , defined according to (2.7), (2.8) makes R diagonal and determines its elements (up to an overall factor) in contrast to the physical case discussed in this paper. Note that the essential difference between \mathbf{h} , generated by (2.6) and $\tilde{\mathbf{h}}$, generated by (4.1) is that in

the first case the unbroken bosonic generators are built from the fermionic while in the second from the bosonic parameters.

Our remark is that $\tilde{\mathbf{h}}$ and this second Yangian play a natural role for the bound states of the *mirror model* [17] obtained by a double Wick rotation from the physical one. The bound states of Q fundamental *mirror* magnons form the $4Q$ dimensional completely *antisymmetric* representation of $su(2|2)$ [17] that can be described in the super-space formalism in terms of the fermionic θ_1, θ_2 and bosonic w_3, w_4 parameters, i.e. in the mirror model one has to make the $w \leftrightarrow \theta$ substitutions. Now making these changes in the generators spanning $su(2|1)$ in (2.6) leads - after the trivial index change $(1, 2) \leftrightarrow (3, 4)$ - to the expressions in (4.1). The fact that the reflection matrix for the $4Q$ dimensional *antisymmetric* representation is diagonal is shown first in a study of finite size effects in boundary AdS/CFT in [21] and its implications for the boundary state of the mirror model are discussed in the Appendix of [22].

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